FINITISTIC DIMENSION THROUGH INFINITE PROJECTIVE DIMENSION.

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Abstract. We show that an artin algebra Λ having at most three radical layers of infinite projective dimension has finite finitistic dimension, generalizing the known result for algebras with vanishing radical cube. We also give an equivalence between the finiteness of $\operatorname{fin.dim.}\Lambda$ and the finiteness of a given class of Λ-modules of infinite projective dimension.

1. Introduction.

Let Λ be an artin algebra, and consider $\operatorname{mod}\Lambda$ the class of finitely generated left Λ-modules. The finitistic dimension of Λ is then defined to be

$$\operatorname{fin.dim.}\Lambda = \sup\{\operatorname{pd} M \ : \ M \in \operatorname{mod}\Lambda \text{ and } \operatorname{pd} M < \infty\},$$

where $\operatorname{pd} M$ denotes the projective dimension of M. It was conjectured by Bass in the 60's that $\operatorname{fin.dim.}\Lambda$ is always finite. Since then, this conjecture was shown to hold for many classes of algebras [2, 7, 4, 9, 10, 11]. In particular the conjecture holds for artin algebras with vanishing radical cube [2, 5, 10]. In this paper, we generalize this result to artin algebras having at most three radical layers of infinite projective dimension.

Theorem A. If ${}_\Lambda\Lambda$ has at most three radical layers of infinite projective dimension, then $\operatorname{fin.dim.}\Lambda$ is finite.

We also provide a bound on the finitistic dimension of Λ under the preceding hypothesis. In order to achieve our goal, we use the Ψ function of Igusa and Todorov [7] and introduce the notion of infinite-layer length which counts in an efficient manner the number of (not necessarily radical) layers of infinite projective dimension of a module. Our next result shows that the finitistic dimension of Λ is in some sense controlled by the syzygies of the simple Λ-modules of infinite projective dimension. Let us denote by $\Omega^n(M)$ the n-th syzygy of M in $\operatorname{mod}\Lambda$.

Theorem B. The following conditions are equivalent for an artin algebra Λ.
(a) The finitistic dimension of Λ is finite,
(b) There exists $s \in \mathbb{N}$ such that $\{\Omega^s(M)\}_{M\in\mathscr{K}(\Lambda)}$ is in $\operatorname{add}\Omega^{s+1}(\Sigma)$,
(c) There exists $s \in \mathbb{N}$ such that $\{\Omega^s(M)\}_{M\in\mathscr{K}(\Lambda)}$ is of finite representation type.

where Σ is the direct sum of all isomorphism classes of simple Λ-modules of infinite projective dimension, and $\mathscr{K}(\Lambda)$ is the class formed by taking all indecomposable M in $\operatorname{mod}\Lambda$ satisfying the following two properties: (a) $\operatorname{pd} M = \infty$ and (b) M is a summand of $\operatorname{rad} L$ for some $L \in \operatorname{mod}\Lambda$ of finite projective dimension.

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2. Notations and definitions.

For an artin algebra Λ and a Λ-module M, we denote by $\operatorname{top} M$ and $\operatorname{soc} M$ the top and socle of M respectively. Given a class $\mathscr{C}$ of objects in $\operatorname{mod}\Lambda$, the projective dimension of $\mathscr{C}$ is $\operatorname{pd}\mathscr{C} := \sup\{\operatorname{pd} M \ : \ M \in \mathscr{C}\}$ if the class $\mathscr{C}$ is not empty, otherwise it is zero. Moreover, the **finitistic dimension** of the class $\mathscr{C}$ is $\operatorname{fin.dim.}\mathscr{C} := \operatorname{pd}\{M \in \mathscr{C} \ : \ \operatorname{pd} M < \infty\}$. We let $\mathcal{S}^{\infty}$ be the finite set consisting of all isomorphism classes of simple Λ-modules of infinite projective dimension. Similarly, we denote by $\mathcal{S}^{<\infty}$ the finite set consisting of all isomorphism classes of simple Λ-modules of finite projective dimension. Then $\alpha := \operatorname{pd}\mathcal{S}^{<\infty}$ is finite. Further, we denote by $[M : \mathcal{S}^{\infty}]$ the number of (not necessarily distinct) composition factors of M of infinite projective dimension. Note that if all the composition factors of M belong to $\mathcal{S}^{<\infty}$ or $M = 0$, then we have $[M : \mathcal{S}^{\infty}] = 0$ and $\operatorname{pd} M \leq \alpha$.

We now recall the definition and main properties of the function Ψ of Igusa and Todorov [7]. Let K denote the quotient of the free abelian group generated by all the symbols $[M]$, where $M \in \operatorname{mod}\Lambda$, by the subgroup generated by symbols of the form: (a) $[C] - [A] - [B]$ if $C \simeq A \oplus B$, and (b) $[P]$ if P is projective. Then K is the free $\mathbb{Z}$-module generated by the iso-classes of indecomposable finitely generated non-projective Λ-modules. In [7], K. Igusa and G. Todorov define the function $\Psi : \operatorname{mod}\Lambda \to \mathbb{N}$ as follows.

The syzygy induces a $\mathbb{Z}$-endomorphism on K that will also be denoted by Ω. That is, we have a $\mathbb{Z}$-homomorphism $\Omega : K \to K$ where $\Omega[M] := [\Omega\, M]$. For a given Λ-module M, we denote by $< M >$ the $\mathbb{Z}$-submodule of K generated by the indecomposable direct summands of M. Since $\mathbb{Z}$ is a Noetherian ring, Fitting's lemma implies that there is an integer n such that $\Omega : \Omega^m < M > \to \Omega^{m+1} < M >$ is an isomorphism for all $m \geq n$; hence there exists a smallest non-negative integer $\Phi\,(M)$ such that $\Omega : \Omega^m < M > \to \Omega^{m+1} < M >$ is an isomorphism for all $m \geq \Phi\,(M)$. Let $\mathscr{C}_M$ be the set whose elements are the direct summands of $\Omega^{\Phi(M)} M$. Then we set:

$$\Psi\,(M) := \Phi\,(M) + \operatorname{fin.dim.}\mathscr{C}_M.$$

The following result is due to K. Igusa and G. Todorov.

Proposition 2.1. [7] *The function* $\Psi : \operatorname{mod}\Lambda \to \mathbb{N}$ *satisfies the following properties.*

(a) *If* $\operatorname{pd}\, M$ *is finite then* $\Psi\,(M) = \operatorname{pd}\, M$. *On the other hand,* $\Psi\,(M) = 0$ *if* M *is indecomposable and* $\operatorname{pd}\, M = \infty$,
(b) $\Psi\,(M) = \Psi\,(N)$ *if* $\operatorname{add}\, M = \operatorname{add}\, N$,
(c) $\Psi(M) \leq \Psi(M \oplus N)$,
(d) $\Psi(M \oplus P) = \Psi(M)$ *for any projective* Λ*-module* P,
(e) *If* $0 \to A \to B \to C \to 0$ *is an exact sequence in* $\operatorname{mod}\Lambda$ *and* $\operatorname{pd}\, C$ *is finite then* $\operatorname{pd}\, C \leq \Psi\,(A \oplus B) + 1$.

Y. Wang proved the following useful inequality, which is a direct consequence of 2.1 (e).

Lemma 2.2. [8] *If* $0 \to A \to B \to C \to 0$ *is an exact sequence in* $\operatorname{mod}\Lambda$ *and* $\operatorname{pd}\, B$ *is finite then* $\operatorname{pd}\, B \leq 2 + \Psi\,(\Omega\, A \oplus \Omega^2\, C)$.

We will also need the following result which appears in our previous paper [6].

Proposition 2.3. [6] *For all* M *in* $\operatorname{mod}\Lambda$, $\Psi\,(M) \leq 1 + \Psi\,(\Omega\, M)$

For a subcategory $\mathscr{C}$ of $\operatorname{mod}\Lambda$, we set $\Psi\text{dim.}\mathscr{C} := \sup\{\Psi(M) \,:\, M \in \mathscr{C}\}$. Since clearly $\Psi\text{dim.}\mathscr{C} < \infty$ implies $\text{fin.dim.}\mathscr{C} < \infty$, in view of the finitistic dimension conjecture, it would be interesting to study this new invariant.

3. The functors Q and S

In this section we introduce endofunctors Q and S on $\operatorname{mod}\Lambda$ that associate to a Λ-module M a quotient $Q(M)$ and a submodule $S(M)$ of M with the properties that the socle of $Q(M)$ and the top of $S(M)$ both lie in $\operatorname{add}\mathcal{S}^{\infty}$ whenever $[M : \mathcal{S}^{\infty}] \neq 0$. These functors will be used in the definition of the infinite-layer length of a module. Throughout this section, we let $\alpha = \operatorname{pd}\mathcal{S}^{<\infty}$.

Given any class $\mathscr{C}$ of Λ-modules, we will consider the full subcategory $\mathcal{F}(\mathscr{C})$ of $\operatorname{mod}\Lambda$ whose objects are the $\mathscr{C}$-filtered Λ-modules. That is, $M \in \mathcal{F}(\mathscr{C})$ if there is a finite chain $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_m = M$ of submodules of M with $m \geq 0$ and such that each quotient M_i/M_{i-1} is isomorphic to some object in $\mathscr{C}$. For example, an object $M \in \operatorname{mod}\Lambda$ is filtered by $\mathcal{S}^{<\infty}$ if and only if $[M : \mathcal{S}^{\infty}] = 0$.

Lemma 3.1. *Let M be a Λ-module. Then the set of all submodules of M filtered by $\mathcal{S}^{<\infty}$ admits a unique maximal element that will be denoted by $K(M)$.*

Proof. Note first that this class is not empty since $[0 : \mathcal{S}^{\infty}] = 0$. Now let K_1 and K_2 be two submodules of M filtered by $\mathcal{S}^{<\infty}$ that are maximal with this property. Since $K_1 + K_2$ is a quotient of $K_1 \oplus K_2 \in \mathcal{F}(\mathcal{S}^{<\infty})$, we get that $K_1 + K_2$ is also filtered by $\mathcal{S}^{<\infty}$. Hence $K_1 = K_1 + K_2 = K_2$. □

Let $Q(M)$ be the quotient $M/K(M)$, that is we have the following exact sequence: $0 \to K(M) \overset{i_M}{\to} M \to Q(M) \to 0$. Given a morphism $f : M \to N$ of Λ-modules, we have the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & K(M) \arrow[r, "i_M"] \arrow[d, dashed, "K(f)"] & M \arrow[r] \arrow[d, "f"] & Q(M) \arrow[r] \arrow[d, dashed, "Q(f)"] & 0 \\
0 \arrow[r] & K(N) \arrow[r, "i_N"] & N \arrow[r] & Q(N) \arrow[r] & 0
\end{tikzcd}$$

Now $fi_M(K(M))$ is a quotient of $K(M) \in \mathcal{F}(\mathcal{S}^{<\infty})$ therefore it is a submodule of N filtered by $\mathcal{S}^{<\infty}$. It then follows from the maximality of $K(N)$ that fi_M factors uniquely through i_N, giving the map $K(f)$. Passing to the cokernels, we obtain a map $Q(f) : Q(M) \to Q(N)$ that makes the above diagram commute. It is then straightforward to verify that K and Q, as defined above, are indeed functors whose main properties are listed below. Note that $\operatorname{pd} K(M) \leq \alpha$.

Proposition 3.2. *The functors $K, Q : \operatorname{mod}\Lambda \to \operatorname{mod}\Lambda$ defined above have the following properties.*

(a) $Q(M) = 0$ *if and only if* $M \in \mathcal{F}(\mathcal{S}^{<\infty})$.

(b) *If* $\operatorname{soc} M \in \operatorname{add}\mathcal{S}^{\infty}$ *then* $Q(M) = M$,

(c) $\operatorname{pd} M < \infty$ *if and only if* $\operatorname{pd} Q(M) < \infty$,

(d) *If* $[M : \mathcal{S}^{\infty}] \neq 0$ *then* $\operatorname{soc} Q(M) \in \operatorname{add}\mathcal{S}^{\infty}$,

(e) $\Omega^{\alpha+1}M \oplus P \cong \Omega^{\alpha+1}Q(M) \oplus P'$ *for some projective Λ-modules P and P'*,

(f) $\operatorname{pd} M \leq \max\{\operatorname{pd} Q(M), \alpha\}$,

(g) *If $f : M \to N$ is a monomorphism (epimorphism), then $Q(f) : Q(M) \to Q(N)$ is a monomorphism (epimorphism),*

(h) $Q^2 = Q$, $K^2 = K$ *and* $K\,Q = 0 = Q\,K$.

Proof. Statements (c), (e) and (f) are easily verified using the exact sequence $0 \to K(M) \to M \to Q(M) \to 0$ and the fact that $\operatorname{pd} K(M) \leq \operatorname{pd} \mathcal{S}^{<\infty} = \alpha < \infty$.

(a) By definition, we have the equivalences $0 = Q(M) \iff K(M) = M \iff [M : \mathcal{S}^{\infty}] = 0$.

(b) If $K(M) \neq 0$, then $0 \neq \operatorname{soc} K(M) \subseteq \operatorname{soc} M \in \operatorname{add} S^{\infty}$, a contradiction since $K(M) \in \mathcal{F}(\mathcal{S}^{<\infty})$. Thus $K(M) = 0$ and $Q(M) = M$.

(d) Since $[M : \mathcal{S}^{\infty}] \neq 0$, we have from (a) that $Q(M) \neq 0$. Assume that $\operatorname{soc} Q(M)$ admits a simple summand S of finite projective dimension, and consider the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & K(M) \arrow[r, "i"] \arrow[d, equal] & E \arrow[r] \arrow[d, "f"] & S \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K(M) \arrow[r] & M \arrow[r] & Q(M) \arrow[r] & 0
\end{tikzcd}$$

where the upper exact sequence is obtained by pullback. Applying the snake lemma, we infer that f is a monomorphism. Moreover, E is filtered by $\mathcal{S}^{<\infty}$ since both $K(M)$ and S are so. The maximality of $K(M)$ then implies that i is an isomorphism, thus $S = 0$, a contradiction.

(g) Let $f : M \to N$ be a morphism of Λ-modules. Consider the following exact and commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & K(M) \arrow[r] \arrow[d, "K(f)"] & M \arrow[r] \arrow[d, "f"] & Q(M) \arrow[r] \arrow[d, "Q(f)"] & 0 \\
0 \arrow[r] & K(N) \arrow[r] & N \arrow[r] & Q(N) \arrow[r] & 0
\end{tikzcd}$$

If $f : M \to N$ is an epimorphism, then from the diagram above, we see that $Q(f)$ is also an epimorphism.

Suppose that $f : M \to N$ is a monomorphism. If $M \in \mathcal{F}(\mathcal{S}^{<\infty})$, then by (a) $Q(M) = 0$; and hence $Q(f)$ is a monomorphism. Assume now that $[M : \mathcal{S}^{\infty}] \neq 0$. It then follows from (d) that $\operatorname{soc} Q(M) \in \operatorname{add} \mathcal{S}^{\infty}$. Applying the snake lemma to the diagram above, we get a monomorphism from $X := \operatorname{Ker} Q(f)$ to $Y := \operatorname{Coker} K(f)$. If $X \neq 0$, then $0 \neq \operatorname{soc} X \subseteq \operatorname{soc} Q(M) \in \operatorname{add} \mathcal{S}^{\infty}$. Also, $\operatorname{soc} X \subseteq \operatorname{soc} Y$, a contradiction since $Y \in \mathcal{F}(\mathcal{S}^{<\infty})$. So $X = 0$ and $Q(f)$ is a monomorphism.

(h) By taking the pull-back to the canonical morphisms $M \to Q(M) \leftarrow K\,Q(M)$, we get the following exact and commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K(M) \arrow[r, "j"] \arrow[d, equal] & E \arrow[r] \arrow[d] & K\,Q(M) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K(M) \arrow[r] & M \arrow[r] \arrow[d] & Q(M) \arrow[r] \arrow[d] & 0 \\
 & & Q^2(M) \arrow[r, equal] \arrow[d] & Q^2(M) \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

Hence, by 3.1 and the fact that $E \in \mathcal{F}(\mathcal{S}^{<\infty})$, we conclude that j is an isomorphism so that $K\,Q(M) = 0$ and $Q(M) = Q^2(M)$. On the other hand, since $K(M) \in \mathcal{F}(\mathcal{S}^{<\infty})$, we get from 3.2(a) that $Q\,K(M) = 0$ and $K^2(M) = K(M)$. $\square$

We now proceed to give the construction of the functor S which is dual to Q. We omit the proofs since they are essentially the same as those we previously did. In what follows, a quotient of M is an epimorphism $g : M \to C$; moreover, if $g_1 : M \to C_1$ and $g_2 : M \to C_2$ are two quotients of M, we say that g_1 is *greater than or equal to* g_2 if there is an epimorphism $h : C_1 \to C_2$ such that $hg_1 = g_2$. Finally, we say that a quotient $g : M \to C$ is filtered by $\mathcal{S}^{<\infty}$ in case C is so.

Lemma 3.3. *Let M be a Λ-module. Then the set of all quotients of M filtered by $\mathcal{S}^{<\infty}$ admits a unique (up to isomorphism) maximal element that will be denoted by $p_M : M \to C(M)$.*

Consider the exact sequence $0 \to S(M) \to M \overset{p_M}{\to} C(M) \to 0$. Given a morphism of Λ-modules $f : M \to N$, we have the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & S(M) \arrow[r] \arrow[d, dashed, "S(f)"] & M \arrow[r, "p_M"] \arrow[d, "f"] & C(M) \arrow[r] \arrow[d, dashed, "C(f)"] & 0 \\
0 \arrow[r] & S(N) \arrow[r] & N \arrow[r, "p_N"] & C(N) \arrow[r] & 0
\end{tikzcd}$$

Now $p_N f(M)$ is a submodule of $C(N) \in \mathcal{F}(\mathcal{S}^{<\infty})$, therefore $M \to p_N f(M)$ is a quotient of M filtered by $\mathcal{S}^{<\infty}$. It then follows from the maximality of $p_M : M \to C_M$ that $p_N f$ factors uniquely through p_M giving us the map $C(f)$. By passing to kernels, we obtain the map $S(f)$ that makes the above diagram commute. It is now straightforward to verify that S and C are functors. We then have the following properties of such functors, the proof of which is dual to 3.2.

Proposition 3.4. *The functors S, $C : \operatorname{mod}\Lambda \to \operatorname{mod}\Lambda$, defined above, have the following properties.*

(a) $S(M) = 0$ *if and only if* $M \in \mathcal{F}(\mathcal{S}^{<\infty})$,

(b) *If* $\operatorname{top} M \in \operatorname{add} S^{\infty}$, *then* $S(M) = M$,

(c) $\operatorname{pd} M < \infty$ *if and only if* $\operatorname{pd} S(M) < \infty$,

(d) *If* $[M : \mathcal{S}^\infty] \neq 0$ *then* $\operatorname{top} S(M) \in \operatorname{add} \mathcal{S}^\infty$,

(e) $\Omega^\alpha M \oplus P \cong \Omega^\alpha S(M) \oplus P'$ *for some projective* Λ*-modules* P *and* P',

(f) $\operatorname{pd} M \leq \max\{\operatorname{pd} S(M), \alpha\}$,

(g) *If* $f : M \to N$ *is a monomorphism (epimorphism), then* $S(f) : S(M) \to S(N)$ *is a monomorphism (epimorphism),*

(h) $C^2 = C$, $S^2 = S$ *and* $C\,S = 0 = S\,C$.

Note that it can also be shown that QS and SQ are naturally isomorphic functors.

4. The infinite-layer length of a module.

In this section, we introduce the invariant $\ell\ell^\infty(M)$ for a Λ-module M. Our goal is to count the number of "layers" of infinite projective dimension of M. A natural way to proceed would be to consider radical layers. Recall that a **radical layer** of a module M is a semisimple module of the form $\operatorname{rad}^i M / \operatorname{rad}^{i+1} M$ for some $0 \leq i < \ell\ell(M)$, where $\ell\ell(M)$ is the Loewy length of M. It then seems reasonable to define the "infinite-layer length" $\ell^\infty(M)$ of a module M as follows.

Definition 4.1. *For any* $M \in \operatorname{mod} \Lambda$, *we set* $\ell^\infty(M)$ *to be the number of radical layers of* M *that have infinite projective dimension.*

As natural as it seems, this definition has some flaws. Firstly, if one proceeds dually and defines $\ell_\infty(M)$ using socle layers, it is not true in general that $\ell^\infty(M) = \ell_\infty(M)$. Also this length does not behave well with submodules. That is, if K is a submodule of M, we do not always have that $\ell^\infty(K) \leq \ell^\infty(M)$.

We will now define another "infinite-layer length" that satisfies the above stated properties and is better than ℓ^∞ in the sense that $\ell\ell^\infty(M) \leq \ell^\infty(M)$. As an analogy, the Loewey length of a module M can be defined to be the smallest nonnegative integer i such that $\operatorname{rad}^i M = 0$. In our case, since we are only interested in layers of infinite projective dimension, we use the functor S to "level" our module prior to taking the radical. This guarantees that at each step, a layer consisting of a maximal number of simples of infinite projective dimension is removed.

Given a module M, we start by calculating $S(M) = M^0$. Unless $S(M) = 0$, the top of M^0 lies in $\operatorname{add} \mathcal{S}^\infty$. We then take the radical of M^0, and let $M^1 := S(\operatorname{rad}(M^0))$. Iterating the process, we let $M^{i+1} := S(\operatorname{rad}(M^i))$ for all $i \geq 0$. This procedure leads us to consider the following additive functor

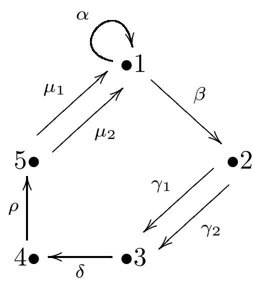
$$F := S \circ \operatorname{rad} : \operatorname{mod} \Lambda \to \operatorname{mod} \Lambda,$$

where S is the functor defined in the previous section and rad is the radical functor. Note that F^0 is the identity functor; and so, by using the functor F, it is now easy to see that $M^i = F^i(S(M))$ for all $i \geq 0$.

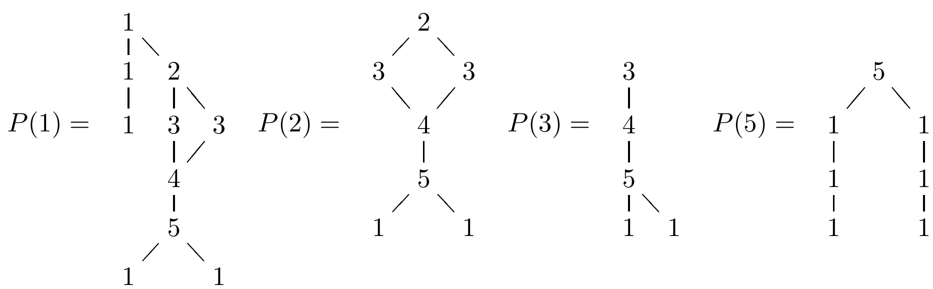
Definition 4.2. *Let* Λ *be an artin algebra. For a finitely generated* Λ*-module* M, *we define the* ***infinite-layer length*** *of* M *to be*

$$\ell\ell^\infty(M) := \min\{i \geq 0 \ : \ F^i(S(M)) = 0\}.$$

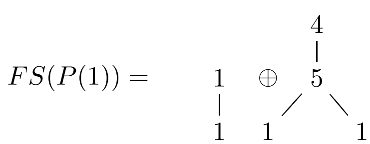
Example 4.3. *Consider the bound quiver algebra $\Lambda = kQ/I$ where Q is given by*



and I is generated by the set of paths $\{\alpha^3, \alpha\beta, \rho\mu_i\alpha, \mu_i\beta, \gamma_1\delta - \gamma_2\delta\}$ where $i = 1, 2$. Then $\mathcal{S}^\infty = \{S(1), S(4)\}$ and the projective Λ-modules are



and $P(4) = \operatorname{rad} P(3)$. In order to compute $\ell\ell^\infty(P(1))$, we need to find the smallest nonnegative integer i such that $F^i(S(P(1))) = 0$. Note that $S(P(1)) = P(1)$ since $\operatorname{top} P(1) = S(1) \in \operatorname{add} \mathcal{S}^\infty$. Thus $FS(P(1)) = F(P(1)) = S\operatorname{rad}(P(1))$ is the following module:



Continuing this process, we obtain $F^2S(P(1)) = (S(1))^3$ and $F^3S(P(1)) = 0$, thus $\ell\ell^\infty(P(1)) = 3$. It is easy to see that among the six radical layers of $P(1)$ only the fifth has finite projective dimension, therefore $\ell^\infty(P(1)) = 5$. Note that $P(1)$ has four socle layers of infinite projective dimension, giving $\ell_\infty(P(1)) = 4$. Further computations show that $\ell\ell^\infty(P(2)) = \ell\ell^\infty(P(3)) = \ell\ell^\infty(P(4)) = 2$ and that $\ell\ell^\infty(P(5)) = 3$.

Lemma 4.4. *Consider the functor $F^i : \operatorname{mod}\Lambda \to \operatorname{mod}\Lambda$ for $i \geq 0$. Then*

(a) *F^i and $F^i \circ S$ preserve monomorphisms and epimorphisms,*

(b) *$F^i\, S(M) \subseteq F^i(M) \subseteq S \circ \operatorname{rad}^i(M)$ for all $M \in \operatorname{mod}\Lambda$ and $i \geq 1$.*

Proof. (a) From 3.4 (g), we know that S preserves both monomorphisms and epimorphisms. This is also true of the functor rad. Consequently, for all non negative integers i, we have that F^i and $F^i \circ S$ also preserves monomorphisms and epimorphisms.

(b) For all $i \geq 0$, we have that $F^i\, S(M) \subseteq F^i(M)$ since $S(M) \subseteq M$ and F^i preserves monomorphisms. The last inclusion will be proved by induction on $i \geq 1$. Firstly, we assert that

$$(*) \qquad \operatorname{rad} \circ S(X) \subseteq \operatorname{rad}(X) \text{ for any } X \in \operatorname{mod} \Lambda.$$

Indeed, since $S(X) \subseteq X$, then $(*)$ follows from the fact that the functor rad preserves monomorphism. We now proceed to the proof of the last inclusion. If $i = 1$ there is nothing to prove. So assume that $i > 1$. Then by hypothesis we have that $F^{i-1}(M) \subseteq S \circ \operatorname{rad}^{i-1}(M)$. Hence, by (a) and $(*)$, we get

$$F^i(M) \subseteq F \circ S \circ \operatorname{rad}^{i-1}(M) = S \circ \operatorname{rad} \circ S(\operatorname{rad}^{i-1}(M)) \subseteq S \circ \operatorname{rad}^i(M).$$

□

The next result shows that $\ell\ell^\infty$ behaves naturally with monomorphisms, epimorphisms and direct sums.

Proposition 4.5. *Let Λ be an artin algebra and L, M and N be Λ-modules.*

(a) *If $f : L \to M$ is a monomorphism, then $\ell\ell^\infty(L) \leq \ell\ell^\infty(M)$,*

(b) *If $g : M \to N$ is an epimorphism, then $\ell\ell^\infty(N) \leq \ell\ell^\infty(M)$,*

(c) *$\ell\ell^\infty(M \oplus N) = \max(\ell\ell^\infty(M), \ell\ell^\infty(N))$,*

(d) *$\ell\ell^\infty(M) \leq \ell\ell^\infty({}_\Lambda\Lambda)$.*

Proof. (a) Let $f : L \to M$ be a monomorphism and assume that $\ell\ell^\infty(M) = m$. Then $F^m(S(M)) = 0$; and so by 4.4(a) we get $F^m(S(L)) = 0$. Thus $\ell\ell^\infty(L) \leq m$, proving (a). A similar argument holds for (b).

(c) Since the functor $F^i\, S$ is aditive, we have that

$$F^i\, S(M \oplus N) \simeq F^i\, S(M) \oplus F^i\, S(N),$$

therefore $\max(\ell\ell^\infty(M), \ell\ell^\infty(N)) \leq \ell\ell^\infty(M \oplus N)$. The reverse inequality follows from the fact that $F^{i_0}S(X) = 0$ implies $F^iS(X) = 0$ for all $i \geq i_0$.

(d) Using that M is a quotient of ${}_\Lambda\Lambda^n$ for some natural number n, we obtain from (b) that $\ell\ell^\infty(M) \leq \ell\ell^\infty({}_\Lambda\Lambda^n)$; thus (d) follows from (c). □

Proposition 4.6. *Let Λ be an artin algebra and M a Λ-module.*

(a) *$\ell\ell^\infty(M) = 0$ if and only if $M \in \mathcal{F}(\mathcal{S}^{<\infty})$,*

(b) *$\ell\ell^\infty(S(M)) = \ell\ell^\infty(M)$,*

(c) *If $\operatorname{top} M$ lies in $\operatorname{add} \mathcal{S}^\infty$ then $\ell\ell^\infty(\operatorname{rad} M) = \ell\ell^\infty(M) - 1$.*

Proof. (a) Let $M \in \mathcal{F}(\mathcal{S}^{<\infty})$. Then by 3.4(a) we have $S(M) = 0$, thus $F^0(S(M)) = 0$ and $\ell\ell^\infty(M) = 0$. Conversely, if $\ell\ell^\infty(M) = 0$ then $S(M) = 0$ and hence $M \in \mathcal{F}(\mathcal{S}^{<\infty})$.

(b) This follows from the definition and the fact that $S^2(M) = S(M)$.

(c) It follows from (a) that $\ell\ell^\infty(M) > 0$ and from 3.4(b) that $S(M) = M$. For all $i \geq 1$, we have $F^i(S(M)) = F^i(M) = (S \circ \operatorname{rad})^{i-1} S\,(\operatorname{rad} M) = F^{i-1}(S(\operatorname{rad} M))$. Therefore $\ell\ell^\infty(\operatorname{rad} M) = \ell\ell^\infty(M) - 1$. □

We now proceed to show that the function $\ell\ell^{\infty}$ behaves well with quotient of socles. This will allow us to show that $\ell\ell^{\infty}$ and its dual $\ell\ell_{\infty}$, which has yet to be defined, are the same invariants.

Proposition 4.7. *Let Λ be an artin algebra and M a Λ-module. If $\operatorname{soc} M \in \operatorname{add} \mathcal{S}^{\infty}$ then $\ell\ell^{\infty}(M/\operatorname{soc} M) = \ell\ell^{\infty}(M) - 1$.*

Proof. Suppose that $\operatorname{soc} M \in \operatorname{add} \mathcal{S}^{\infty}$ and let $m = \ell\ell^{\infty}(M) > 0$. We claim that $N := F^{m-1}(S(M)) \subseteq \operatorname{soc} M$. Indeed, from 4.4(a), we have $0 \neq N \subseteq M$ and from the definition of $\ell\ell^{\infty}$, we have $S(\operatorname{rad}(N)) = F^m(S(M)) = 0$. In particular, $0 \neq \operatorname{soc} N \subseteq \operatorname{soc} M \in \operatorname{add} \mathcal{S}^{\infty}$. Now if N is not semisimple, then $\operatorname{rad} N \neq 0$. Therefore, we get from 3.4(a) that $S(\operatorname{rad}(N)) \neq 0$ since $0 \neq \operatorname{soc}(\operatorname{rad} N) \subseteq \operatorname{soc} N \in \operatorname{add} \mathcal{S}^{\infty}$. This is a contradiction, hence N is semisimple and $N = \operatorname{soc} N$, proving that $N \subseteq \operatorname{soc} M$.

Consider the projection $p : M \to M/\operatorname{soc} M$. Then by 4.4(a) the map

$$q = F^{m-1}S(p) : F^{m-1}S(M) \to F^{m-1}S(M/\operatorname{soc} M)$$

is also an epimorphism and is actually the restriction of p to $F^{m-1}S(M)$. We therefore have the following commutative diagram

$$\begin{tikzcd} F^{m-1}S(M) \arrow[r, "q"] \arrow[d, "i"] & F^{m-1}S(M/\operatorname{soc} M) \arrow[d, "j"] \\ M \arrow[r, "p"] & M/\operatorname{soc} M \end{tikzcd}$$

where i and j are the inclusions. From what precedes, we know that $F^{m-1}S(M) \subseteq \operatorname{soc} M$, thus $jq = pi = 0$ and hence $q = 0$, implying that $F^{m-1}S(M/\operatorname{soc} M) = 0$ and $\ell\ell^{\infty}(M/\operatorname{soc} M) \leq m-1$. Assume now that $\ell\ell^{\infty}(M/\operatorname{soc} M) = l < m-1$. Then by replacing $m-1$ for l in the above diagram, we get $0 = jq = pi$ hence $F^l S(M) \subseteq \operatorname{soc} M$. But then $0 = (S \circ \operatorname{rad})F^l S(M) = F^{l+1}S(M)$ gives us $\ell\ell^{\infty}(M) \leq l + 1 < m$, a contradiction. □

We can now define the dual of $\ell\ell^{\infty}$. The strategy is the same as before except that we now proceed from bottom to top using the functor Q. The definition is slightly more tedious since we have to proceed with quotients by socles instead of radicals. Given a module M, we start by calculating $M_0 := Q(M)$. Unless $Q(M) = 0$, the socle of M_0 lies in $\operatorname{add} \mathcal{S}^{\infty}$. We then let $M_1 := Q(M_0/\operatorname{soc} M_0)$. Iterating the process, we let $M_{i+1} := Q(M_i/\operatorname{soc} M_i)$ for all $i \geq 0$. This procedure leads us to consider the additive functor

$$G := Q/(\operatorname{soc} \circ Q) : \operatorname{mod} \Lambda \to \operatorname{mod} \Lambda,$$

where Q is the functor defined in 3.2 and soc is the socle functor. Then, by using the functor G, it is now easy to see that $M_i = Q \circ G^i(M)$ for all integers $i \geq 0$.

Definition 4.8. *Let Λ be an artin algebra. For a finitely generated Λ-module M, we define*

$$\ell\ell_{\infty}(M) := \min\{i \geq 0 \ : \ Q \circ G^i(M) = 0\}.$$

Example 4.9. *We calculate $\ell\ell_{\infty}(P(1))$ from example 4.3. Recall that $\mathcal{S}^{\infty} = \{S(1), S(4)\}$ hence $\operatorname{soc} P(1) \in \operatorname{add} \mathcal{S}^{\infty}$. Therefore $QG^0(P(1)) = Q(P(1)) = P(1)$ and $QG(P(1)) = Q((P(1)/\operatorname{soc} P(1))$ is the following module*

$$QG(P(1)) = \begin{array}{ccc} 1 & & \\ | \diagdown & & \\ 1 & 2 & \\ & | \diagdown & \\ & 3 & 3 \\ & | \diagup & \\ & 4 & \end{array}$$

Continuing this process, we get that $QG^2(P(1)) = S(1)$ and that $QG^3(P(1)) = 0$, thus $\ell\ell_{\infty}(P(1)) = 3$.

Proposition 4.10. *Let Λ be an artin algebra and M be a finitely generated Λ-module. Then*

(a) $\ell\ell_{\infty}(Q(M)) = \ell\ell_{\infty}(M)$,

(b) *If $\operatorname{soc}(M) \in \operatorname{add} \mathcal{S}^{\infty}$, then $\ell\ell_{\infty}(M/\operatorname{soc} M) = \ell\ell_{\infty}(M) - 1$,*

(c) *$\ell\ell^{\infty}(D(M)) = \ell\ell_{\infty}(M)$, where $D : \operatorname{mod} \Lambda \to \operatorname{mod} \Lambda^{op}$ is the usual duality functor.*

Proof. (a) This follows from the definition and the fact that $Q^2(M) = Q(M)$.

(b) We know from 3.2(b) that $Q(M) = M$. Therefore, we obtain from the definition that $M_1 = Q(Q(M)/\operatorname{soc} Q(M)) = Q(M/\operatorname{soc} M) = (M/\operatorname{soc} M)_0$ and hence $M_{i+1} = (M/\operatorname{soc} M)_i$ for all $i \geq 0$, giving $\ell\ell_{\infty}(M/\operatorname{soc} M) = \ell\ell_{\infty}(M) - 1$.

(c) In order to prove (c), we need to show that $D(M_i) \simeq (D(M))^i$ for all $i \geq 0$ (see in 4.2). We will do it by induction on i. Applying the duality functor D to the exact sequence $0 \to K(M) \to M \to Q(M) \to 0$, and utilizing the fact that $DK(M)$ is the maximal quotient of $D(M)$ filtered by $D(\mathcal{S}^{\infty})$ together with 3.3, we get an isomorphism $DQ(M) \simeq SD(M)$ which is natural on M; proving the statement for $i = 0$. For the inductive step, we have that

$$\begin{aligned} D(M_{i+1}) &= DQ(M_i/\operatorname{soc} M_i) \\ &\simeq SD(M_i/\operatorname{soc} M_i) \\ &\simeq S\operatorname{rad}(DM_i) \\ &\simeq F(D(M_i)) \\ &\simeq F(DQ(M_i)) \\ &\simeq F(SD(M_i)) \\ &= FS(D(M))^i \\ &\simeq (D(M))^{i+1}, \end{aligned}$$

since $D(M/\operatorname{soc} M) \simeq \operatorname{rad}(DM)$ and $M_i = Q\, M_i$. $\square$

We are now in position to show that $\ell\ell_{\infty}$ and $\ell\ell^{\infty}$ coincide.

Theorem 4.11. *Let Λ be an artin algebra. If M is a finitely generated Λ-module, then*

$$\ell\ell^{\infty}(M) = \ell\ell_{\infty}(M).$$

Proof. Firstly, we will show by induction on $\ell\ell^{\infty}(M)$ that $\ell\ell_{\infty}(M) \leq \ell\ell^{\infty}(M)$. If $\ell\ell^{\infty}(M) = 0$ then $S(M) = 0$. Thus by 3.4(a), we have $[M : \mathcal{S}^{\infty}] = 0$. This implies, by 3.2(a), that $Q(M) = 0$ and then $\ell\ell_{\infty}(M) = 0$.

Suppose that $\ell\ell^{\infty}(M) \geq 1$. Since $Q(M)$ is a quotient of M we know from 4.5(b) that $\ell\ell^{\infty}(Q(M)) \leq \ell\ell^{\infty}(M)$. Therefore, by 3.2(d) and 4.7 we obtain

$$\ell\ell^{\infty}(Q(M)/\operatorname{soc} Q(M)) = \ell\ell^{\infty}(Q(M)) - 1 \leq \ell\ell^{\infty}(M) - 1 < \ell\ell^{\infty}(M),$$

and by induction, we infer that $\ell\ell_{\infty}(Q(M)/\operatorname{soc} Q(M)) \leq \ell\ell^{\infty}(Q(M)/\operatorname{soc} Q(M))$. On the other hand, it follows from 4.10(a) that $\ell\ell_{\infty}(Q(M)) = \ell\ell_{\infty}(M)$. Thus, by 4.10(b), 3.2(d) and 4.7, letting $G(M) = Q(M)/\operatorname{soc} Q(M)$, we get

$$\ell\ell_{\infty}(M) = \ell\ell_{\infty}(G(M)) + 1 \leq \ell\ell^{\infty}(G(M)) + 1 = \ell\ell^{\infty}(Q(M)) \leq \ell\ell^{\infty}(M).$$

Finally, we prove that $\ell\ell_{\infty}(M) \geq \ell\ell^{\infty}(M)$. Indeed, using 4.10(c), we have

$$\ell\ell^{\infty}(M) = \ell\ell_{\infty}(DM) \leq \ell\ell^{\infty}(DM) = \ell\ell_{\infty}(M),$$

giving us the desired equality. $\square$

Corollary 4.12. *Let Λ be an artin algebra and M a finitely generated Λ-module. Then*

(a) $\ell\ell^{\infty}(Q(M)) = \ell\ell^{\infty}(M)$,

(b) *if* $\ell\ell^{\infty}(M) > 0$ *then* $\ell\ell^{\infty}(Q(M)/\operatorname{soc} Q(M)) = \ell\ell^{\infty}(M) - 1$.

Proof. (a) This follows from 4.10(a) and 4.11. On the other hand, (b) is a consequence of (a), 3.2(d) and 4.7. $\square$

We conclude this section by giving the precise relation between ℓ^{∞} and $\ell\ell^{\infty}$. Recall that $\ell^{\infty}(M)$ is the number of radical layers of M of infinite projective dimension. We will construct a nonnegative function r with the property that for all $M \in \operatorname{mod}\Lambda$, $\ell^{\infty}(M) = \ell\ell^{\infty}(M) + r(M)$. This implies that for all M, $\ell^{\infty}(M) \geq \ell\ell^{\infty}(M)$ a fact that will be used in the next section.

Following 3.4(a), we have that $\mathcal{S}^{\infty}$ is not empty if and only if the class $\{M \in \operatorname{mod}\Lambda \,:\, S(M) \neq 0\}$ is not empty. This will be utilized in the construction of the following function.

Lemma 4.13. *Let Λ be an artin algebra such that $\mathcal{S}^{\infty} \neq \emptyset$. Then there exists a unique function $\varphi : \{M \in \operatorname{mod}\Lambda \,:\, S(M) \neq 0\} \longrightarrow \mathbb{N}$ satisfying the following properties:*

(a) $S(M) = S \circ \operatorname{rad}^{j}(M)$ *for all* $0 \leq j \leq \varphi(M)$,

(b) $\operatorname{pd}(\operatorname{top}(\operatorname{rad}^{\varphi(M)}(M))) = \infty$,

(c) $\operatorname{pd}(\operatorname{top}(\operatorname{rad}^{i}(M))) < \infty$ *for any* $0 \leq i < \varphi(M)$.

Proof. Let $M \in \operatorname{mod}\Lambda$ be such that $S(M) \neq 0$. If $\operatorname{pd}(\operatorname{top} M) = \infty$, we set $\varphi(M) := 0$. If $\operatorname{pd}(\operatorname{top} M) < \infty$, we have that $\operatorname{top} M \in \mathcal{F}(\mathcal{S}^{<\infty})$; and hence, by 3.3, we get the following exact commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & S(M) \arrow[r] \arrow[d, dashed, "i"] & M \arrow[r] \arrow[d, equal] & C(M) \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & \operatorname{rad}(M) \arrow[r] & M \arrow[r] & \operatorname{top} M \arrow[r] & 0
\end{tikzcd}$$

where i is the inclusion map. Then, by 3.4(g) and (h), we conclude that

$$S(M) = S \circ \mathrm{rad}\,(M).$$

If $\mathrm{pd}\,(\mathrm{top}\,(\mathrm{rad}\,(M))) = \infty$, we set $\varphi(M) := 1$, otherwise we would have that $\mathrm{top}\,(\mathrm{rad}\,(M)) \in \mathcal{F}\,(\mathcal{S}^{<\infty})$; and hence, we can repeat the same procedure by replacing M by $\mathrm{rad}\,(M)$ in the diagram above. Therefore, we get in this case that $S \circ \mathrm{rad}\,(M) = S \circ \mathrm{rad}^{\,2}(M)$. So we have

$$0 \neq S(M) = S \circ \mathrm{rad}\,(M) = S \circ \mathrm{rad}^{\,2}(M).$$

Since the functor $\mathrm{rad} : \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Lambda$ is nilpotent and $0 \neq S(M)$, it is clear that this procedure has to finish. That is, there exists a smallest positive integer $\varphi(M)$ such that $\mathrm{pd}\,(\mathrm{top}\,(\mathrm{rad}^{\,\varphi(M)}(M))) = \infty$, proving the result. $\square$

Next, we construct a function that counts radical layers of infinite projective dimension. We use the following notation for intervals of natural numbers: $[i,j]_{\mathbb{N}} := \{x \in \mathbb{N} \;:\; i \leq x \leq j\}$, $[i,j)_{\mathbb{N}} := \{x \in \mathbb{N} \;:\; i \leq x < j\}$ and the other obvious possibilities. Note that if $\ell\ell^{\,\infty}(M) > 0$ then $F^n\,S(M) \neq 0$ for all $n \in [0, \ell\ell^{\,\infty}(M))_{\mathbb{N}}$.

Theorem 4.14. *Let Λ be an artin algebra and $M \in \mathrm{mod}\,\Lambda$. If $\ell\ell^{\,\infty}(M) > 0$, then there exists a function $\zeta = \zeta_M : [0, \ell\ell^{\,\infty}(M))_{\mathbb{N}} \longrightarrow [0, \ell\ell\,(M))_{\mathbb{N}}$ satisfying the following properties:*

(a) *For any $i \in [0, \ell\ell^{\,\infty}(M))_{\mathbb{N}}$, we have that*

  (a1) *$F^i\,S(M) \subseteq S \circ \mathrm{rad}^{\,\zeta(i)}(M)$, $i \leq \zeta(i)$, and*

  (a2) *$\mathrm{pd}\,(\mathrm{top}\,(\mathrm{rad}^{\,\zeta(i)}(M))) = \infty$,*

(b) *$\mathrm{pd}\,(\mathrm{top}\,(\mathrm{rad}^{\,j}(M))) < \infty$ for any $j \in [0, \zeta(\ell\ell^{\,\infty}(M) - 1)]_{\mathbb{N}} - \mathrm{Im}(\zeta)$,*

(c) *the function ζ can be computed by the following recurrence relation*

$$\zeta(0) = \varphi(M),$$
$$\zeta(i+1) = \zeta(i) + 1 + \varphi(\mathrm{rad}^{\,\zeta(i)+1}(M)).$$

Proof. Let $M \in \mathrm{mod}\,\Lambda$ be such that $\ell\ell^{\,\infty}(M) > 0$. By 4.13, we have that $F^0 S(M) = S \circ \mathrm{rad}^{\,\varphi(M)}(M)$; and so, we set $\zeta(0) := \varphi(M)$. In order to define $\zeta(1)$, we proceed as follows. Firstly, we apply 4.4(b) to the above equality obtaining

$$0 \neq FS(M) = F\,S(\mathrm{rad}^{\,\zeta(0)}(M)) \subseteq S \circ \mathrm{rad}^{\,\zeta(0)+1}(M).$$

Note that, now, we can apply 4.13 to $\mathrm{rad}^{\,\zeta(0)+1}(M)$. Then we have

$$S \circ \mathrm{rad}^{\,\zeta(0)+1}(M) = S \circ \mathrm{rad}^{\,\varphi(\mathrm{rad}^{\,\zeta(0)+1}(M))}(\mathrm{rad}^{\,\zeta(0)+1}(M)).$$

Therefore, $\zeta(1) := \zeta(0) + 1 + \varphi(\mathrm{rad}^{\,\zeta(0)+1}(M))$ satifies the required properties. Suppose that $\zeta(i)$ has been defined. To define $\zeta(i+1)$, we apply 4.4(b) to the inclusion $F^i\,S(M) \subseteq S \circ \mathrm{rad}^{\,\zeta(i)}(M)$, obtaining that

$$0 \neq F^{i+1}S(M) = F\,S(\mathrm{rad}^{\,\zeta(i)}(M)) \subseteq S \circ \mathrm{rad}^{\,\zeta(i)+1}(M).$$

So, we can apply 4.13 to $\mathrm{rad}^{\,\zeta(i)+1}(M)$. Then we have

$$S \circ \mathrm{rad}^{\,\zeta(i)+1}(M) = S \circ \mathrm{rad}^{\,\varphi(\mathrm{rad}^{\,\zeta(i)+1}(M))}(\mathrm{rad}^{\,\zeta(i)+1}(M)).$$

Hence, $\zeta(i+1) := \zeta(i) + 1 + \varphi(\mathrm{rad}^{\zeta(i)+1}(M))$ satifies the required properties, proving the result. $\square$

For any $M \in \mathrm{mod}\,\Lambda$ we set $r^\infty(M) = 0$ in case $\ell\ell^\infty(M) = 0$, otherwise $r^\infty(M)$ is equal to the cardinality of the set

$$\{k \in \mathbb{N} \ : \ k > \zeta(\ell\ell^\infty(M) - 1) \text{ and } \mathrm{pd}\,(\mathrm{top}\,(\mathrm{rad}^k(M))) = \infty\}.$$

We also recall that $\ell^\infty(M)$ denotes the number of radical layers of M of infinite projective dimension.

Corollary 4.15. *If Λ is an artin algebra and $M \in \mathrm{mod}\,\Lambda$, then*

$$\ell\ell^\infty(M) + r^\infty(M) = \ell^\infty(M).$$

Proof. If $\ell\ell^\infty(M) = 0$, we have, by 4.6 (a), that $[M : \mathcal{S}^\infty] = 0$. Hence M contains no radical layer of infinite projective dimension so that $\ell^\infty(M) = 0$. So, we may assume that $\ell\ell^\infty(M) > 0$; and then, the result follows from 4.14. $\square$

Note that, in general, it could happen that $r^\infty(M) > 0$, as can be seen in the example given in the previous section.

5. Main results.

We now proceed to show that an artin algebra with at most three radical layers of infinite projective dimension has finite finitistic dimension. In section two, we introduced $\Psi\mathrm{dim}.\mathscr{C}$ for a given subcategory $\mathscr{C}$ of $\mathrm{mod}\,\Lambda$. We now show that this invariant is finite for certain subclasses of $\mathrm{mod}\,\Lambda$. Throughout this section, we will use the notation $\alpha = \mathrm{pd}\,\mathcal{S}^{<\infty}$ and $\Sigma = \oplus_{S\in\mathcal{S}^\infty} S$.

Definition 5.1. *Let Λ be an artin algebra and M a finitely generated Λ-module such that $[M : \mathcal{S}^\infty] \neq 0$. Then, we set*

$$S_M^\infty := \bigoplus_{S\in\mathcal{S}^\infty} S^{[M:S]} \in \mathrm{add}\,\mathcal{S}^\infty.$$

Lemma 5.2. *Let Λ be an artin algebra and M a finitely generated Λ-module. If $\ell\ell^\infty(M) = 1$, then*

(a) $S_M^\infty = S_{Q(M)}^\infty = \mathrm{soc}\,Q(M)$ *and* $\mathrm{pd}\,M = \infty$,

(b) $\Omega^{\alpha+1}(M) \oplus P \simeq \Omega^{\alpha+1}(S_M^\infty) \oplus P'$ *for some projective Λ-modules P and P'.*

Proof. Suppose that $\ell\ell^\infty(M) = 1$. Then 4.12 (b) gives $\ell\ell^\infty(Q(M)/\mathrm{soc}\,Q(M)) = 0$; and so, by 4.6(a), we conclude that $Q(M)/\mathrm{soc}\,Q(M) \in \mathcal{F}(\mathcal{S}^{<\infty})$. Therefore, by considering the following exact sequence

$$0 \to \mathrm{soc}\,Q(M) \to Q(M) \to Q(M)/\mathrm{soc}\,Q(M) \to 0,$$

we obtain that $\mathrm{soc}\,Q(M)$ contains all composition factors of $Q(M)$ (and hence of M) of infinite projective dimension. This proves the first part of (a) since $\mathrm{soc}\,Q(M) \in \mathrm{add}\,\mathcal{S}^\infty$ (see 3.2(d)).

On the other hand, since $\mathrm{pd}\,Q(M)/\mathrm{soc}\,Q(M) \leq \alpha$ and $\mathrm{pd}\,\mathrm{soc}\,Q(M) = \infty$, we have $\mathrm{pd}\,Q(M) = \infty$ and hence from 3.2(c), $\mathrm{pd}\,M = \infty$. Also, the above exact sequence yields $\Omega^{\alpha+1}Q(M) \simeq \Omega^{\alpha+1}\mathrm{soc}\,Q(M)$. Thus (b) follow from 3.2(e). $\square$

Definition 5.3. *Let Λ be an artin algebra. For each $i \in [0, \ell\ell^{\infty}({}_{\Lambda}\Lambda)]_{\mathbb{N}}$, we introduce the following class*

$$\mathscr{L}_i^{\infty} = \mathscr{L}_i^{\infty}(\Lambda) := \{M \in \operatorname{mod} \Lambda : \ell\ell^{\infty}(M) \leq i\}.$$

Observe that 4.6(a) gives us the equality $\mathscr{L}_0^{\infty} = \mathcal{F}(\mathcal{S}^{<\infty})$. Therefore, from 2.1(a), we get $\operatorname{fin.dim.} \mathscr{L}_0^{\infty} = \Psi\operatorname{dim.} \mathscr{L}_0^{\infty} = \alpha$. It is natural to ask if $\Psi\operatorname{dim.} \mathscr{L}_i^{\infty}$ is finite for all i, or if $\operatorname{fin.dim.} \mathscr{L}_i^{\infty}$ is finite for all i. A positive answer to either question would imply the validity of the finitistic dimension conjecture since from 4.5(d), the classes $\mathscr{L}_i^{\infty}$ give rise to a finite filtration of $\operatorname{mod} \Lambda$

$$\mathscr{L}_0^{\infty} \subseteq \mathscr{L}_1^{\infty} \subseteq \cdots \subseteq \mathscr{L}_{i_0}^{\infty} = \operatorname{mod} \Lambda,$$

where $i_0 = \ell\ell^{\infty}({}_{\Lambda}\Lambda)$.

The next result partially answers the question raised above. In some sense, it also generalizes the results of [6]. Recall that $\alpha = \operatorname{pd} \mathcal{S}^{<\infty}$ and $\Sigma = \oplus_{S \in \mathcal{S}^{\infty}} S$.

Proposition 5.4. *Let Λ be an artin algebra. Then*

(a) $\Psi\operatorname{dim.} \mathscr{L}_1^{\infty}(\Lambda) \leq \alpha + 1 + \Psi(\Omega^{\alpha+1}(\Sigma))$,

(b) $\operatorname{fin.dim.} \mathscr{L}_2^{\infty}(\Lambda) \leq \alpha + 2 + \Psi(\Omega^{\alpha+1}(\Sigma) \oplus \Omega^{\alpha+2}(\Sigma))$.

Proof. (a) Let M be in $\mathscr{L}_1^{\infty}(\Lambda)$. If $\ell\ell^{\infty}(M) = 0$, it follows from 4.6(a) that $\Psi\operatorname{dim.} M = \operatorname{pd} M \leq \alpha$. If $\ell\ell^{\infty}(M) = 1$, we have from 5.2(b) that

$$\Omega^{\alpha+1}(M) \oplus P \simeq \Omega^{\alpha+1}(S_M^{\infty}) \oplus P'$$

for some projective Λ-modules P and P'. Applying 2.3 as well as 2.1(b),(c) and (d), we have

$$\Psi(M) \leq \alpha + 1 + \Psi(\Omega^{\alpha+1}(M)) = \alpha + 1 + \Psi(\Omega^{\alpha+1}(S_M^{\infty})) \leq \alpha + 1 + \Psi(\Omega^{\alpha+1}(\Sigma));$$

proving (a).

(b) Let $M \in \mathscr{L}_2^{\infty}(\Lambda)$ be of finite projective dimension. Then by 3.2(c), $\operatorname{pd} Q(M)$ is finite and by 5.2(a), $\ell\ell^{\infty}(M)$ is either equal to 0 or 2. If $\ell\ell^{\infty}(M) = 0$, it follows from 4.6(a) that $\operatorname{pd} M \leq \alpha$. Assume that $\ell\ell^{\infty}(M) = 2$. Applying 4.12(b), we obtain $\ell\ell^{\infty}(G(M)) = 1$. Hence, from 5.2(b), we get

$$\Omega^{\alpha+1}(G(M)) \oplus P_0 \simeq \Omega^{\alpha+1}(S_{G(M)}^{\infty}) \oplus P_0'$$

for some projective Λ-modules P_0 and P_0'. Applying 2.2 and 2.3 to the exact sequence $0 \to \operatorname{soc} Q(M) \to Q(M) \to G(M) \to 0$, we get

$$\begin{array}{rcl}
\operatorname{pd} Q(M) & \leq & 2 + \Psi[\Omega(\operatorname{soc} Q(M)) \oplus \Omega^2(G(M))] \\
& \leq & 2 + \alpha + \Psi[\Omega^{\alpha+1}(\operatorname{soc} Q(M)) \oplus \Omega^{\alpha+2}(G(M))] \\
& = & 2 + \alpha + \Psi[\Omega^{\alpha+1}(\operatorname{soc} Q(M)) \oplus \Omega^{\alpha+2}(S_{G(M)}^{\infty})] \\
& \leq & 2 + \alpha + \Psi[\Omega^{\alpha+1}(\Sigma) \oplus \Omega^{\alpha+2}(\Sigma)],
\end{array}$$

where the last two inequalities follow from 2.1 (c) and (b) respectively. It then follows from 3.2 (f) that

$$\operatorname{pd} M \leq \max\{\operatorname{pd} Q(M), \alpha\} \leq 2 + \alpha + \Psi(\Omega^{\alpha+1}(\Sigma) \oplus \Omega^{\alpha+2}(\Sigma)),$$

proving the result. $\square$

Lemma 5.5. *Let Λ be an artin algebra such that $\mathcal{S}^{\infty} \neq \emptyset$, and $M \in \operatorname{mod} \Lambda$, then*

$$\ell\ell^{\infty}(\Omega S(M)) \leq \ell\ell^{\infty}({}_{\Lambda}\Lambda) - 1.$$

Proof. Let $M \in \operatorname{mod}\Lambda$. If $[M : \mathcal{S}^\infty] = 0$, then we have by 3.4(a) that $\Omega S(M) = 0$, and $\ell\ell^\infty(0) = 0 \leq \ell\ell^\infty({}_\Lambda\Lambda) - 1$ since $\mathcal{S}^\infty \neq \emptyset$.

Suppose that $[M : \mathcal{S}^\infty] \neq 0$ and let P be the projective cover of $S(M)$. On one hand, we have from 4.5(d) that $\ell\ell^\infty(P) - 1 \leq \ell\ell^\infty({}_\Lambda\Lambda) - 1$. On the other hand, using 3.4(d), we get that $\operatorname{top} P \simeq \operatorname{top} S(M) \in \operatorname{add} \mathcal{S}^\infty$; thus, by 4.6(c), we have $\ell\ell^\infty(\operatorname{rad} P) = \ell\ell^\infty(P) - 1$. Finally, since $\Omega S(M) \subseteq \operatorname{rad} P$, we obtain using 4.5(a) that $\ell\ell^\infty(\Omega S(M)) \leq \ell\ell^\infty(\operatorname{rad} P)$ proving the result. □

We are now in position to show our first main result. Note that if Λ is an artin algebra such that $\mathcal{S}^\infty = \emptyset$, then $\operatorname{gl.dim}\Lambda = \operatorname{fin.dim.}\Lambda = \alpha$. In what follows, we let $\beta := \{\ell\ell^\infty({}_\Lambda\Lambda) - 1\}$.

Theorem 5.6. *Let Λ be an artin algebra such that $\mathcal{S}^\infty \neq \emptyset$, then*

$$\operatorname{fin.dim.}\Lambda \leq \max\{\alpha, 1 + \operatorname{fin.dim.} \mathscr{L}^\infty_\beta(\Lambda)\}.$$

Proof. Let $M \in \operatorname{mod}\Lambda$ be of finite projective dimension. Then we have from 3.4(c) that $\operatorname{pd} S(M)$ is also finite. Using 3.4(f), we have $\operatorname{pd} M \leq \max\{\alpha, \operatorname{pd} S(M)\} \leq \max\{\alpha, 1 + \operatorname{pd} \Omega S(M)\}$. Now from 5.5, $\Omega S(\operatorname{mod}\Lambda) \subseteq \mathscr{L}^\infty_\beta(\Lambda)$, yielding the result. □

Theorem 5.7. *Let Λ be an artin algebra. If $\ell\ell^\infty({}_\Lambda\Lambda) \leq 3$, then*

$$\operatorname{fin.dim.}\Lambda \leq \alpha + 3 + \Psi(\Omega^{\alpha+1}(\Sigma) \oplus \Omega^{\alpha+2}(\Sigma)) < \infty.$$

Proof. Let $\ell\ell^\infty({}_\Lambda\Lambda) \leq 3$. If $\mathcal{S}^\infty = \emptyset$, then $\operatorname{fin.dim.}\Lambda = \alpha$. Otherwise, $\beta \leq 2$ and the result follows from 5.4(b) and 5.6. □

Example 5.8. *Consider the algebra Λ of example 4.3. We showed in this example that*

$$\max\{\ell\ell^\infty(P(i)) : 1 \leq i \leq 5\} = 3,$$

therefore we infer from 4.5(c) that $\ell\ell^\infty({}_\Lambda\Lambda) = 3$. Also, $\mathcal{S}^\infty = \{S(1), S(4)\}$ and $\alpha = \operatorname{pd}\mathcal{S}^{<\infty} = 2$. Using 2.1 we get $\Psi[\Omega^3(S(1) \oplus S(4)) \oplus \Omega^4(S(1) \oplus S(4))] = \Psi(S(1) \oplus T) = 0$, where T is the two-dimensional indecomposable module whose top and socle are both isomorphic to $S(1)$. It then follows from 5.7 that $\operatorname{fin.dim.}\Lambda \leq 5$.

Corollary 5.9. *If Λ is an artin algebra such that ${}_\Lambda\Lambda$ has at most three radical layers of infinite projective dimension, then* $\operatorname{fin.dim.}\Lambda$ *is finite.*

Proof. By hypothesis and 4.15, we have $\ell\ell^\infty({}_\Lambda\Lambda) \leq \ell^\infty({}_\Lambda\Lambda) \leq 3$. Thus, the result follows from 5.7. □

The technique used in 5.4 to prove that $\operatorname{fin.dim.} \mathscr{L}^\infty_2(\Lambda)$ is finite does not apply directly to the general case. The crucial point of the proof of 5.4(b) is that the Λ-module $G(M)$ with the property $\ell\ell^\infty(G(M)) = 1$ satisfies $\Omega^{\alpha+1}(G(M)) \oplus P \simeq \Omega^{\alpha+1}(S^\infty_{G(M)}) \oplus P'$ for some projective Λ-modules P and P'. Unfortunately, we do not have this type of isomorphism for an arbitrary module M of infinite projective dimension with $\ell\ell^\infty(M) = i > 1$. We now define two classes of modules that have the above mentioned behavior. In order to do that, we denote by $\operatorname{ind}^\infty(\Lambda)$ the class

of all indecomposable Λ-modules of infinite projective dimension, and by $\mathcal{P}^{<\infty}(\Lambda)$ the class of all finitely generated Λ-modules having finite projective dimension.

Definition 5.10. *For an artin algebra Λ, we set*

(a) $\mathscr{C}(\Lambda) = \mathrm{ind}^{\infty}(\Lambda) \cap \{\mathrm{add}\,(M/\mathrm{soc}\,M) \,:\, M \in \mathcal{P}^{<\infty}(\Lambda)\}$,

(b) $\mathscr{K}(\Lambda) = \mathrm{ind}^{\infty}(\Lambda) \cap \{\mathrm{add}\,(\mathrm{rad}\,M) \,:\, M \in \mathcal{P}^{<\infty}(\Lambda)\}$.

We now introduce a function that will be used in the theorem below. Under certain hypotheses, this function will provide a bound for $\mathrm{pd}\,M$ depending on $\ell\ell^{\infty}(M)$ when $M \in \mathcal{P}^{<\infty}(\Lambda)$.

Definition 5.11. *Let Λ be an artin algebra. Suppose that there is a natural number $s \in \mathbb{N}$ and a finitely generated Λ-module L such that $\Omega^s(\mathscr{C}(\Lambda)) \subseteq \mathrm{add}\,L$. Each natural number $t \in \mathbb{N}$ induces a function $[t]_L : \mathbb{N} \to \mathbb{N}$ that can be computed by the following recurrence relation*

$$[t]_L(0) := t,$$
$$[t]_L(i+1) := [t]_L(i) + 2 + s + \Psi(\Omega^{1+s+[t]_L(i)}(\Sigma) \oplus \Omega^{2+[t]_L(i)}(L)).$$

It is not difficult to see that $[t]_L(i+1) = [[t]_L(i)]_L(1)$ for all $i \geq 0$.

Lemma 5.12. *Let Λ be an artin algebra such that there is an $s \in \mathbb{N}$ and a finitely generated Λ-module L satisfying $\Omega^s(\mathscr{C}(\Lambda)) \subseteq \mathrm{add}\,L$. For any $M \in \mathcal{P}^{<\infty}(\Lambda)$ we have*

(a) *$G(M) = \mathsf{C}^{\infty}(M) \oplus \mathsf{C}^{<\infty}(M)$ with $\mathsf{C}^{\infty}(M) \in \mathrm{add}\,\mathscr{C}(\Lambda)$ and $\mathsf{C}^{<\infty}(M) \in \mathcal{P}^{<\infty}(\Lambda)$;*

(b) *if $[M : \mathcal{S}^{\infty}] \neq 0$ and $\mathrm{pd}\,\mathsf{C}^{<\infty}(M) \leq t$, then*
$$\mathrm{pd}\ M \leq \max(\alpha, [t]_L(1)) \ \text{ and } \ \ell\ell^{\infty}(\mathsf{C}^{<\infty}(M)) \leq \ell\ell^{\infty}(M) - 1.$$

Proof. By 3.2(c), we know that $Q(M) \in \mathcal{P}^{<\infty}(\Lambda)$. So (a) follows from the exact sequence $0 \to \mathrm{soc}\,Q(M) \to Q(M) \to G(M) \to 0$ and the definition of $\mathscr{C}(\Lambda)$.

Suppose that $[M : \mathcal{S}^{\infty}] \neq 0$ and $\mathrm{pd}\,\mathsf{C}^{<\infty}(M) \leq t$. Then applying 2.2 and 3.2(d) to the above exact sequence, we get

$$\mathrm{pd}\ Q(M) \leq 2 + \Psi(\Omega(\mathrm{soc}\,Q(M)) \oplus \Omega^2 G(M)) \ \text{ and } \ \mathrm{soc}\,Q(M) \in \mathrm{add}\,\Sigma.$$

On the other hand, since $\Omega^{2+t}G(M) = \Omega^{2+t}\mathsf{C}^{\infty}(M)$ and $\Omega^s(\mathscr{C}(\Lambda)) \subseteq \mathrm{add}\,L$, we conclude that $\Omega^{2+s+t}G(M) = \Omega^{2+t}(X)$ for some $X \in \mathrm{add}\ L$. Hence, by 2.3 and 2.1, we obtain

$$\mathrm{pd}\ Q(M) \leq 2 + t + s + \Psi(\Omega^{1+s+t}(\Sigma) \oplus \Omega^{2+t}(L)) = [t]_L(1).$$

Hence, the first inequality in (b) follows from 3.2(f). The last inequality in (b) follows from 4.5(a) since, by 4.12(b), we know that $\ell\ell^{\infty}(G(M)) = \ell\ell^{\infty}(M) - 1$. □

Theorem 5.13. *Let Λ be an artin algebra such that there is an $s \in \mathbb{N}$ and a finitely generated Λ-module L satisfying that $\Omega^s(\mathscr{C}(\Lambda)) \subseteq \mathrm{add}\,L$. Then*

(a) *$\mathrm{pd}\ M \leq [\alpha]_L(\ell\ell^{\infty}(M))$ for all $M \in \mathcal{P}^{<\infty}(\Lambda)$,*

(b) *$\mathrm{fin.dim.}\ \Lambda \leq [\alpha]_L(\ell\ell^{\infty}({}_{\Lambda}\Lambda)) < \infty$.*

Proof. (a) Let $M \in \mathcal{P}^{<\infty}(\Lambda)$. We will apply induction on $\ell\ell^{\infty}(M)$. If $\ell\ell^{\infty}(M) = 0$ then $M \in \mathcal{F}(\mathcal{S}^{<\infty})$; and so pd $M \leq$ pd $\mathcal{S}^{<\infty} = \alpha = [\alpha]_L(0)$.

Suppose that $\ell\ell^{\infty}(M) \geq 1$. In particular, from 4.6(a), we have $[M : \mathcal{S}^{\infty}] \neq 0$. Therefore we conclude from 5.12(b) that $\ell\ell^{\infty}(\mathsf{C}^{<\infty}(M)) \leq \ell\ell^{\infty}(M) - 1$. Hence, by induction, we have that pd $\mathsf{C}^{<\infty}(M) \leq [\alpha]_L(\ell\ell^{\infty}(\mathsf{C}^{<\infty}(M)))$. Then, applying again 5.12(b) with $t := [\alpha]_L(\ell\ell^{\infty}(\mathsf{C}^{<\infty}(M)))$, we get

$$\text{pd } M \leq \max(\alpha, [t]_L(1)).$$

On the other hand, since $[t]_L(i+1) = [[t]_L(i)]_L(1)$ and $[t]_L(i+1) > [t]_L(i)$, we have $[t]_L(1) = [[\alpha]_L(\ell\ell^{\infty}(\mathsf{C}^{<\infty}(M)))]_L(1) = [\alpha]_L(\ell\ell^{\infty}(\mathsf{C}^{<\infty}(M)) + 1) \leq [\alpha]_L(\ell\ell^{\infty}(M))$, proving pd $M \leq [\alpha]_L(\ell\ell^{\infty}(M))$.

(b) We know from 4.5(d) that $\ell\ell^{\infty}(M) \leq \ell\ell^{\infty}({}_\Lambda\Lambda)$. Thus from (a) we have pd $M \leq [\alpha]_L(\ell\ell^{\infty}(M)) \leq [\alpha]_L(\ell\ell^{\infty}({}_\Lambda\Lambda))$ for all $M \in \mathcal{P}^{<\infty}(\Lambda)$. □

The "dual" version of the above result is easily proved replacing $Q(M)$ by $S(M)$, socles by tops and $G(M)$ by $F(M)$.

Theorem 5.14. *Let Λ be an artin algebra such that there is an $s \in \mathbb{N}$ and a finitely generated Λ-module L satisfying $\Omega^s(\mathscr{K}(\Lambda)) \subseteq \operatorname{add} L$. Then*

(a) pd $M \leq \langle\alpha\rangle_L(\ell\ell^{\infty}(M))$ *for any* $M \in \mathcal{P}^{<\infty}(\Lambda)$,

(b) fin.dim. $\Lambda \leq \langle\alpha\rangle_L(\ell\ell^{\infty}({}_\Lambda\Lambda)) < \infty$;

where the function $\langle\alpha\rangle_L : \mathbb{N} \to \mathbb{N}$ can be computed by the following recurrence relation

$$\langle\alpha\rangle_L(0) := \alpha := \text{pd } \mathcal{S}^{<\infty},$$
$$\langle\alpha\rangle_L(i+1) := \langle\alpha\rangle_L(i) + 2 + s + \Psi(\Omega^{2+s+\langle\alpha\rangle_L(i)}(\Sigma) \oplus \Omega^{1+\langle\alpha\rangle_L(i)}(L)).$$

We prove now that the conditions we assumed in the two results above are rather general. In fact such conditions are equivalent to the finiteness of fin.dim. Λ. We then have our main result

Theorem 5.15. *Let Λ be an artin algebra and let Σ be the direct sum of all iso-classes of simple Λ-modules of infinite projective dimension. The following conditions are equivalent:*

(a) *The finitistic dimension of Λ is finite,*

(b) *There exists $s \in \mathbb{N}$ such that $\Omega^{s+1}(\mathscr{C}(\Lambda)) \subseteq \operatorname{add}(\Omega^s(\Sigma))$,*

(c) *There exists $s \in \mathbb{N}$ and $L \in \operatorname{mod}\Lambda$ such that $\Omega^s(\mathscr{C}(\Lambda)) \subseteq \operatorname{add} L$,*

(d) *There exists $s \in \mathbb{N}$ such that $\Omega^s(\mathscr{K}(\Lambda)) \subseteq \operatorname{add}(\Omega^{s+1}(\Sigma))$.*

(e) *There exists $s \in \mathbb{N}$ and $L \in \operatorname{mod}\Lambda$ such that $\Omega^s(\mathscr{K}(\Lambda)) \subseteq \operatorname{add} L$.*

Proof. We first prove that (a) implies (b). Assume that fin.dim. Λ is finite. Let $r \geq 1 + \text{fin.dim. } \Lambda$ and $C \in \mathscr{C}(\Lambda)$. By definition we have an exact sequence

$$0 \to \operatorname{soc} M \to M \to C \oplus X \to 0$$

with pd M finite and pd $\operatorname{soc} M = \infty$. Therefore, since $\Omega^r(\mathcal{S}^{<\infty}) = 0$, we get that $\Omega^r(\operatorname{soc} M) = \Omega^r(S^{\infty}_{\operatorname{soc} M}) \in \operatorname{add} \Omega^r(\Sigma)$. On the other hand, from the above exact sequence, we conclude that

$$\Omega^{r+1}(C \oplus X) \oplus P \simeq \Omega^r(\operatorname{soc} M) \oplus P'$$

for some projective Λ-modules P and P'. Letting $s = r+1$, we obtain $\Omega^{s+1}(C) \in \operatorname{add}(\Omega^s(\Sigma))$.

(b)⇒(c) is trivial, and (c)⇒(a) follows from 5.13. The equivalences of (a), (d) and (e) can be done similarly by using 5.14. □

A class $\mathscr{C} \in \operatorname{mod}\Lambda$ is said to be of finite representation type if there exists a finitely generated Λ-module L such that $\mathscr{C} \subseteq \operatorname{add} L$. We can thus reformulate 5.15.

Theorem 5.16. *For an artin algebra* Λ*,* fin.dim. Λ *is finite if and only if* $\Omega^s(\mathscr{C}(\Lambda))$ *is of finite representation type for some* $s \geq 0$*.*

In [3] Coehlo showed that an artin algebra with finitely many isoclasses of indecomposable of infinite projective dimension has finite finitistic dimension. This was done using the fact that, in this case, $\mathcal{P}^{<\infty}(\Lambda)$ is contravariantly finite (see [1]). Since $\mathscr{C}(\Lambda) \subseteq \operatorname{ind}^{\infty}(\Lambda)$, we can use 5.16 (with $s = 0$) to obtain an alternate proof of this result.

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